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Steady rotation of a tethered sphere at small, non-zero Reynolds and Taylor numbers: wake interference effects on drag

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Matched asymptotic expansion methods are used to solve the title problem. First-order Taylor number corrections to both the Stokes-law drag and Kirchhoff's-law couple on the sphere are obtained for Rossby numbers of order unity. This calculation fills a gap between the Proudman–Pearson (1957) rectilinear trajectory analysis, which includes Reynolds-number effects but does not address Taylor-number effects arising from the curvilinear trajectory, and the Herron, Davis & Bretherton (1975) curvilinear-trajectory analysis, which incorporates Taylor-number effects but ignores those arising from the Reynolds number. At the same Reynolds number, the drag on the sphere is found to be greater or less than the classical Oseen (1927)–Proudman & Pearson (1957) value, depending upon the magnitude of a certain dimensionless length parameter B measuring the tether radius to the sphere radius. This drag difference is attributed, in part, to the fact that the sphere runs into the disturbance created by its own wake.

1. Introduction

This paper addresses the steady rotation of a tethered sphere through a viscous fluid at small, non-zero Taylor and Reynolds numbers, the Rossby number being assumed of order unity. Since the sphere-centre trajectory is closed upon itself, the sphere continuously moves through the residual disturbance left by its own wake. Of interest is the question of whether, in consequence of this phenomenon, the drag on the sphere differs from that which would obtain if (at the same translational Reynolds number) its path were rectilinear rather than curvilinear. In rectilinear motion the wake always lies aft of the sphere.

Of course, no difference is possible in the quasi-static Stokes limit, where all (density-induced) wake phenomena are necessarily absent. Only when fluid inertial and/or Coriolis and/or centrifugal effects are sensible are such differences possible. Our goal is to quantify this difference when such density-derived effects, though small, are non-zero. As discussed at the conclusion of this paper, our tethered-sphere analysis may be applied essentially unchanged to the 'antisedimentation' rotating

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tube device of Dill & Brenner (1983) and Nadim, Cox & Brenner (1985). Indeed, it was to aid in the further development of this device that the present analysis was undertaken.

1.1. *Historical review*

As an historical footnote (cf. Prandtl & Tietjens 1934; von Kármán 1954), identical drag questions – though in the opposite limit of *large* Reynolds numbers – arose in the very early days of drag (and lift) measurements. Many such tethered-body experiments were performed, with the bodies under study mounted at the end (or opposite ends) of a whirling arm. Such rotary-arm 'towing' devices were widely used prior to the advent of wind tunnels in the first decades of the present century. Von Kármán (1954) and Rouse & Ince (1957) trace the use of such whirling-arm devices as far back as Benjamin Robins (1707–1751) and Jean Charles de Borda (1733–1799), though the wake disturbance effect upon the hydrodynamic resistance of the body† was apparently not articulated until the drag and lift experiments of the aeronautical pioneer Otto Lilienthal in 1870 (Lilienthal 1889). Apparently to avoid such curvilinear wake interference problems (and in the transition period just prior to the successful development of high-speed wind tunnels as viable alternatives to whirling arm devices), the civil engineer Alexandre Gustave Eiffel (1832–1923) used his famous turn-of-the-century Paris Exhibition tower (Eiffel 1907, 1910) to perform free-fall rectilinear drag and lift measurements on variously shaped bodies. Thus, excepting the Reynolds-number-range involved, the tethered body-wake interference problem to be addressed here can be said to possess a long and auspicious history. Despite this, we are not aware of any pertinent theoretical studies of the matter.

1.2. *Relevant literature*

The slow motion of a sphere in a viscous rotating fluid has been studied by several authors. Childress (1964) and, more recently, Weisenborn (1985) simplified the inertial terms by restricting the sphere to move along the axis of rotation. Herron, Davis & Bretherton (1975) assumed the sphere velocity relative to the fluid to be small enough to neglect inertia terms in the rotating set of axes. Drew (1978) eliminated the centrifugal terms by imposing the rotation as a condition at infinity. Each of these authors calculated first-order corrections to the Stokes drag and Kirchhoff couple on the sphere by using three-dimensional Fourier transforms to construct the velocity field due a Stokeslet at the origin, subsequently calculating the additional velocities engendered by the higher-order terms in the Navier–Stokes equations. Existing experimental work on the subject is reviewed and correlated by Karanfilian & Kotas (1981).

Many other analytical and numerical investigations of the combined effects of simultaneous translational and rotational particle motions immediately beyond the Stokes range exist (Rubinow & Keller 1961; Cox 1965; Singh 1975*a*,*b*; Dennis, Ingham & Singh 1982), but none of these are strictly pertinent to the title problem.

1.3. *Problem outline*

The fluid motion engendered by a tethered sphere whose centre traverses a circle at uniform speed in a fluid at rest at infinity will be supposed steady relative to rotating axes. Coriolis, inertial and centrifugal terms in the rotating-frame Navier–Stokes

† Even greater complications were created by the fact that the supporting arm itself tended to set the surrounding fluid into synchronous rotation, while simultaneously experiencing a retarding force above and beyond the drag force acting on the tethered body.

equations, are assumed to be scaled such that each is of the same order as the viscous and pressure terms in the outer field, while being negligible in the inner region. This is tantamount to an Oseen-type linearization in the outer flow field (Drew 1978), though in fact the actual asymptotic analysis is performed systematically by matched asymptotic expansions rather than *ad hoc* Oseen linearization.

As in the case of the well-known rectilinear Oseen (1927) wake behind the body, some form of circular wake can be anticipated. Naturally, the latter structure features prominently in the mathematical analysis, in which the coordinate-system origin is chosen to lie on the axis of rotation – in contrast with the analyses of Childress (1964), Herron *et al.* (1975) and Drew (1978). Moreover, Fourier series and Hankel transforms (in the plane of the sphere's motion) are used in place of the Fourier transforms employed by these three authors. Having obtained the velocity and pressure fields in this manner, the drag, lift and torque modifications derived by Herron *et al.* (1975) of the conventional quasi-static Stokes results are conveniently subtracted out, and the residual non-Stokesian contributions to the hydrodynamic force and torque expressed as triple integrals. These are then evaluated numerically, and the drag result compared with the rectilinear Oseen (1927)–Proudman & Pearson (1957) Reynolds-number correction to Stokes law.

2. Basic equations

Referring to figure 1, consider a freely rotating sphere whose centre C is constrained to rotate steadily about a fixed point O with angular velocity vector $\boldsymbol{\Omega}$. We propose to calculate the hydrodynamic force $\boldsymbol{F}'$ and torque $\boldsymbol{M}'$ (about C) exerted by the fluid on the sphere in the limit

$$\frac{a}{b} \ll 1, \tag{2.1}$$

and for small, but non-zero, Reynolds and Taylor numbers – in a precise sense to be defined.† Because of the circular trajectory of the sphere, the latter runs into the disturbance created by its own wake. Consequently, for a prescribed sphere-centre speed $U = \Omega b$, the drag on the sphere is expected (and found) to be different from that which would obtain if the sphere followed a rectilinear rather than a curvilinear path. Calculation of this difference is the principal goal of this paper, bearing in mind that the disparity must necessarily disappear in the Stokes, zero-inertial-effect limit.

In general, the equations of motion of an incompressible Newtonian fluid from the viewpoint of an observer fixed in the rotating system are (Greenspan 1968)

$$\frac{\partial \boldsymbol{v}'}{\partial t} + \boldsymbol{v}' \cdot \boldsymbol{\nabla}' \boldsymbol{v}' + 2\boldsymbol{\Omega} \times \dot{\boldsymbol{v}}' + \dot{\boldsymbol{\Omega}} \times \boldsymbol{R}' = -\frac{1}{\rho} \boldsymbol{\nabla}' p' + \boldsymbol{g}' + \nu \boldsymbol{\nabla}'^2 \boldsymbol{v}' \tag{2.2}$$

and
$$\boldsymbol{\nabla}' \cdot \boldsymbol{v}' = 0, \tag{2.3}$$

with
$$p' = \pi' + P', \tag{2.4}$$

in which
$$P' = -\tfrac{1}{2}[\Omega^2 R'^2 - (\boldsymbol{\Omega} \cdot \boldsymbol{R}')^2] \tag{2.5}$$

is the fictitious fluid pressure due to the centrifugal force on the fluid, and π' the true fluid-mechanical pressure. Here, $\boldsymbol{R}'$ is the position vector of a fluid point measured from an origin lying along the axis of rotation. The external body force $\boldsymbol{g}'$ due to

† The Rossby number (Greenspan 1968; Batchelor 1967), defined as $U/\Omega L$, with L here taken to be the radius b of the circular trajectory, is of order unity in the subsequent theory.

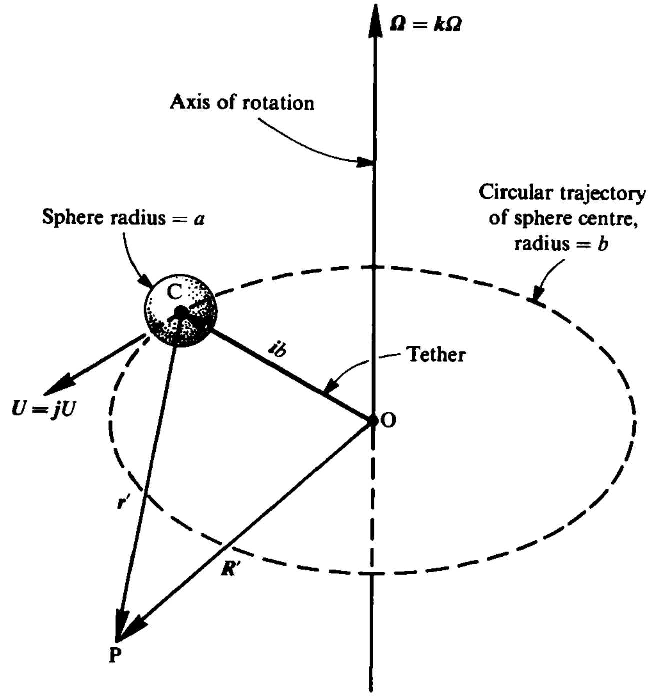

FIGURE 1. Tethered sphere rotating steadily about an asymmetric axis. The angular velocity of rotation is $\boldsymbol{\Omega} = \boldsymbol{k}\Omega$, with $\Omega = |\boldsymbol{\Omega}|$. The sphere is of radius a and its centre C is situated at a distance b from the axis of rotation. With O a point on the axis of rotation lying in the same plane as the sphere centre, we have that $\overrightarrow{\mathrm{OC}} = \boldsymbol{i}b$, with $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ right-handed Cartesian unit vectors fixed in the rotating system. The coordinates of a fluid point P are, respectively, $\boldsymbol{R}'$ and $\boldsymbol{r}'$ with respect to origins at O and C – these two position vectors being related as in (2.9). Thus, the sphere surface $|\boldsymbol{r}'| = a$ becomes $|\boldsymbol{R}' - \boldsymbol{i}b| = a$ in the $\boldsymbol{R}'$ system. Relative to a stationary observer, the instantaneous velocity $\boldsymbol{U} = \boldsymbol{\Omega} \times \boldsymbol{i}b$ of the sphere centre is $\boldsymbol{U} = \boldsymbol{j}U$, where $U = \Omega b$. Relative to this same observer, the angular velocity of the sphere about an axis through its centre C is $\boldsymbol{\Omega}$.

gravity may, as usual, be absorbed into p', and we shall suppose without further comment that in what follows this has been done.

The velocity $\boldsymbol{q}'$ measured by an Earth-fixed, non-rotating observer is related to $\boldsymbol{v}'$ by the expression

$$\boldsymbol{q}' = \boldsymbol{v}' + \boldsymbol{\Omega} \times \boldsymbol{R}'. \tag{2.6}$$

The boundary condition that the fluid be at rest at infinity relative to the stationary observer requires that

$$\boldsymbol{q}' \to \boldsymbol{0} \quad \text{as } |\boldsymbol{R}'| \to \infty, \tag{2.7}$$

whereas the condition that the sphere neither translate nor rotate from the vantage point of the rotating observer requires that

$$\boldsymbol{v}' = \boldsymbol{0} \quad \text{on the sphere surface } |\boldsymbol{r}'| = a, \tag{2.8}$$

in which, referring to figure 1,

$$\boldsymbol{r}' = \boldsymbol{R}' - \boldsymbol{i}b. \tag{2.9}$$

In (2.2) we are interested only in the case of steady rotation ($\dot{\boldsymbol{\Omega}} = \boldsymbol{0}$), and then only in the relative steady state $\partial \boldsymbol{v}'/\partial t = \boldsymbol{0}$ eventually attained after any initial transients decay. In such circumstances, the basic equations of motion and boundary conditions, expressed entirely in terms of the velocity $\boldsymbol{q}'$ recorded by the stationary observer, become

$$\boldsymbol{\Omega}\times\boldsymbol{q}'+(\boldsymbol{q}'-\boldsymbol{\Omega}\times\boldsymbol{R}')\cdot\nabla'\boldsymbol{q}' = -\rho^{-1}\nabla' p'+\upsilon\nabla'^2\boldsymbol{q}', \tag{2.10}$$

$$\nabla'\cdot\boldsymbol{q}' = 0, \tag{2.11}$$

$$\boldsymbol{q}'\to\boldsymbol{0} \quad \text{as } |\boldsymbol{R}'|\to\infty \tag{2.12}$$

and

$$\boldsymbol{q}' = \boldsymbol{\Omega}\times\boldsymbol{R}' \quad \text{on } |\boldsymbol{R}'-\boldsymbol{i}b| = a, \tag{2.13}$$

wherein $\nabla' \equiv \partial/\partial\boldsymbol{R}' \equiv \partial/\partial\boldsymbol{r}'$.

3. Near field

3.1. *Inner variables and equations*

Define non-dimensional inner variables by the relations

$$\boldsymbol{r}' = a\boldsymbol{r}, \quad \nabla' \equiv \frac{\partial}{\partial\boldsymbol{r}'} = a^{-1}\nabla, \quad \boldsymbol{q}' = U\boldsymbol{q}, \quad p'-p'_\infty = \rho\upsilon a^{-1}Up, \tag{3.1}$$

wherein $\nabla \equiv \partial/\partial\boldsymbol{r}$, and in which

$$U = \Omega b \tag{3.2}$$

is the speed of the sphere centre relative to a stationary observer, while p'_∞ is the uniform pressure at infinity. Additionally, define the Taylor and Reynolds numbers (based upon sphere radius) as

$$T = \frac{\Omega a^2}{\upsilon} \ll 1 \tag{3.3}$$

and

$$Re = \frac{Ua}{\upsilon} \ll 1. \tag{3.4}$$

In terms of these new variables, the exact governing equations and boundary conditions (2.10)–(2.13) become

$$T\boldsymbol{k}\times\boldsymbol{q}+Re\left(\boldsymbol{q}-\boldsymbol{j}-\frac{a}{b}\,\boldsymbol{k}\times\boldsymbol{r}\right)\cdot\nabla\boldsymbol{q} = -\nabla p+\nabla^2\boldsymbol{q}, \tag{3.5}$$

$$\nabla\cdot\boldsymbol{q} = 0, \tag{3.6}$$

$$\boldsymbol{q}\to\boldsymbol{0} \quad \text{as } |\boldsymbol{r}|\to\infty, \tag{3.7}$$

$$\boldsymbol{q} = \boldsymbol{j}+\frac{a}{b}\,\boldsymbol{k}\times\boldsymbol{r} \quad \text{on } |\boldsymbol{r}| = 1. \tag{3.8}$$

Following Childress (1964) in the solution of a related problem, we will suppose that†

$$Re = T^{\frac{1}{2}}B = O(T^{\frac{1}{2}}). \tag{3.9}$$

Explicitly, the path radius b is assumed to be comparable with the Eckman length $(\upsilon/\Omega)^{\frac{1}{2}}$ because

$$B \overset{\text{def}}{=} b\left(\frac{\Omega}{\upsilon}\right)^{\frac{1}{2}} = O(1). \tag{3.10}$$

† The two other possible limiting cases, $Re \gg T^{\frac{1}{2}}$ and $Re \ll T^{\frac{1}{2}}$, respectively correspond to those already treated by Proudman & Pearson (1957) and Herron *et al.* (1975).

Note that this makes

$$\frac{a}{b} = \frac{T^{\frac{1}{2}}}{B}, \tag{3.11}$$

which, since we have supposed $T \ll 1$ and $B = O(1)$, is consistent with our assumption (2.1).

With use of (3.10) and (3.11), the basic inner equations (3.5) and (3.8) become, exactly,

$$T[\boldsymbol{k} \times \boldsymbol{q} - (\boldsymbol{k} \times \boldsymbol{r}) \cdot \nabla \boldsymbol{q}] + T^{\frac{1}{2}} B(\boldsymbol{q} - \boldsymbol{j}) \cdot \nabla \boldsymbol{q} = -\nabla p + \nabla^2 \boldsymbol{q} \tag{3.12}$$

and

$$\boldsymbol{q} = \boldsymbol{j} + \frac{T^{\frac{1}{2}}}{B} \boldsymbol{k} \times \boldsymbol{r} \quad \text{on } |\boldsymbol{r}| = 1, \tag{3.13}$$

to which (3.6) and (3.7) are to be appended. Note that the rotational terms $\boldsymbol{k} \times \boldsymbol{r}$ are $O(T^{\frac{1}{2}})$ smaller than their translational counterparts $\boldsymbol{j}$.

3.2. *Inner expansion*

Following Herron *et al.* (1975) we seek inner expansions of the forms

$$\boldsymbol{q}(\boldsymbol{r}; T) = \boldsymbol{q}_0(\boldsymbol{r}) + T^{\frac{1}{2}} \boldsymbol{q}_1(\boldsymbol{r}) + O(T), \tag{3.14}$$

$$p(\boldsymbol{r}; T) = p_0(\boldsymbol{r}) + T^{\frac{1}{2}} p_1(\boldsymbol{r}) + O(T), \tag{3.15}$$

for the solution of the above exact system of inner equations.

3.3. *Zero-order inner field*

The zero-order inner fields correspond to the classic Stokes flow of a sphere of unit radius translating uniformly (with velocity $\boldsymbol{j}$) in a fluid of unit viscosity that is at rest at infinity. The well-known solution of this problem is (Lamb 1932)

$$\boldsymbol{q}_0 = \left[\frac{3}{4r}(\boldsymbol{I} + \hat{\boldsymbol{r}}\hat{\boldsymbol{r}}) + \frac{1}{4r^3}(\boldsymbol{I} - 3\hat{\boldsymbol{r}}\hat{\boldsymbol{r}})\right] \cdot \boldsymbol{j} \tag{3.16}$$

and

$$p_0 = \frac{3}{2r^2} \hat{\boldsymbol{r}} \cdot \boldsymbol{j}. \tag{3.17}$$

Here, $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$ is a unit radial vector. The non-dimensional force and torque on the sphere (about its centre) arising from this motion are

$$\boldsymbol{f}_0 = -6\pi \boldsymbol{j}, \tag{3.18}$$

$$\boldsymbol{m}_0 = \boldsymbol{0}. \tag{3.19}$$

Upon converting (3.16), (3.17) to outer independent and dependent variables via (4.6) below, we find that the matching condition requires that the outer fields satisfy

$$\boldsymbol{Q}_0 \sim \frac{3}{4|\boldsymbol{R} - \boldsymbol{i}B|} \left[\boldsymbol{I} + \frac{(\boldsymbol{R} - \boldsymbol{i}B)(\boldsymbol{R} - \boldsymbol{i}B)}{|\boldsymbol{R} - \boldsymbol{i}B|^2}\right] \cdot \boldsymbol{j}, \tag{3.20}$$

$$P_0 \sim \frac{3}{2} \frac{(\boldsymbol{R} - \boldsymbol{i}B)}{|\boldsymbol{R} - \boldsymbol{i}B|^3} \cdot \boldsymbol{j} \tag{3.21}$$

as $|\boldsymbol{R} - \boldsymbol{i}B| \to 0$. This field pair represents the classical Stokeslet solution (Lamb 1932) for a point force of strength $-6\pi \boldsymbol{j}$ situated at the origin, $\boldsymbol{R} = \boldsymbol{i}B$.

4. Far field

4.1. *Outer variables and equations*

Define non-dimensional outer variables by the equations

$$\boldsymbol{R}' = \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\boldsymbol{R}, \quad \nabla' \equiv \frac{\partial}{\partial \boldsymbol{R}'} = \left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}}\nabla_R, \quad \boldsymbol{q}' \doteq UT^{\frac{1}{2}}\boldsymbol{Q}, \quad p'-p'_\infty = \rho\nu a^{-1}UTP, \quad (4.1)$$

in which $\nabla_R \equiv \partial/\partial\boldsymbol{R}$. In terms of these variables, the exact governing equations and boundary conditions (2.10)–(2.13) become

$$\boldsymbol{k}\times\boldsymbol{Q}+T^{\frac{1}{2}}B\boldsymbol{Q}\cdot\nabla_R\boldsymbol{Q}-(\boldsymbol{k}\times\boldsymbol{R})\cdot\nabla_R\boldsymbol{Q} = -\nabla_R P+\nabla_R^2\boldsymbol{Q}, \quad (4.2)$$

$$\nabla_R\cdot\boldsymbol{Q} = 0, \quad (4.3)$$

$$\boldsymbol{Q}\to\boldsymbol{0} \quad \text{as } |\boldsymbol{R}|\to\infty, \quad (4.4)$$

$$\boldsymbol{Q} = \frac{1}{T^{\frac{1}{2}}B}\boldsymbol{k}\times\boldsymbol{R} \quad \text{on } |\boldsymbol{R}-\boldsymbol{i}B| = 1. \quad (4.5)$$

In the second term appearing on the left-hand side of (4.2) we have replaced the coefficient *Re* that would otherwise have appeared by $T^{\frac{1}{2}}B$, in accordance with (3.9). Equations (4.2)–(4.5) constitute the exact system of outer equations.

We note from our definitions that corresponding inner and outer variables are related by

$$\boldsymbol{q} = T^{\frac{1}{2}}\boldsymbol{Q}, \quad p = TP, \quad \boldsymbol{r} = T^{-\frac{1}{2}}(\boldsymbol{R}-\boldsymbol{i}B). \quad (4.6a,b,c)$$

Assume outer expansions of the forms

$$\boldsymbol{Q}(\boldsymbol{R};T) = \boldsymbol{Q}_0(\boldsymbol{R})+T^{\frac{1}{2}}\boldsymbol{Q}_1(\boldsymbol{R})+O(T), \quad (4.7)$$

$$P(\boldsymbol{R};T) = P_0(\boldsymbol{R})+T^{\frac{1}{2}}P_1(\boldsymbol{R})+O(T). \quad (4.8)$$

Substitute these into (4.2)–(4.4) and equate terms of equal orders in T; however, this procedure is inapplicable to the boundary condition (4.5) on the sphere surface since the outer expansion is expected to be valid only in the outer region $|\boldsymbol{R}-\boldsymbol{i}B| = O(1)$. Standard matching techniques show that $\boldsymbol{Q}_0$ must satisfy an inner boundary condition corresponding to a point force of strength $-6\pi\boldsymbol{j}$ situated at the point $\boldsymbol{R}-\boldsymbol{i}B = \boldsymbol{0}$ where the sphere centre is located. Hence the equations for $\boldsymbol{Q}_0$ are

$$\nabla_R^2\boldsymbol{Q}_0-\nabla_R P_0-\boldsymbol{k}\times\boldsymbol{Q}_0+(\boldsymbol{k}\times\boldsymbol{R})\cdot\nabla_R\boldsymbol{Q}_0 = -\boldsymbol{j}6\pi\delta(\boldsymbol{R}-\boldsymbol{i}B), \quad (4.9)$$

$$\nabla_R\cdot\boldsymbol{Q}_0 = 0, \quad (4.10)$$

$$\boldsymbol{Q}_0\to\boldsymbol{0} \quad \text{as } |\boldsymbol{R}|\to\infty, \quad (4.11)$$

with δ the Dirac delta function.

Equation (4.9) may be regarded as a classical Oseen (1927) linearization of the Navier–Stokes equations (2.10), achieved by employing the far-field, Oseen-type approximation $\boldsymbol{v}' \sim -\boldsymbol{\Omega}\times\boldsymbol{R}'$, i.e. $\boldsymbol{q}' \sim \boldsymbol{0}$ (derived from (2.6) and (2.7)) as the coefficient of the inertial term $\boldsymbol{q}'\cdot\nabla\boldsymbol{q}'$ in (2.10). In this sense, the interrelation existing between the Stokes solution, given by (3.16), (3.17), and the 'Oseen' solution of (4.9)–(4.11), including the matching condition, is precisely the same as in the classical Proudman & Pearson (1957) resolution of the translational counterpart of our rotational problem.

4.2. *Zero-order outer field*

In contrast to the work of Herron *et al.* (1975), terms of order $T^{-\frac{1}{2}}$ do not appear in the expansions (4.7) and (4.8) because they have been anticipated by working with $\boldsymbol{q}$ instead of $\boldsymbol{v}$ (the latter being the dimensionless counterpart of $\boldsymbol{v}'$ appearing in (2.2)). Those authors solved (4.9) in the presence of the Coriolis term, but in the absence of the inertial term, by employing a three-dimensional Fourier transform. However, this method is less helpful now because the additional terms $(X\,\partial Q_0/\partial Y - Y\,\partial Q_0/\partial X)$ – with $\boldsymbol{R} \equiv (X, Y, Z)$ – cause the appearance, in the transformed equations, of derivatives with respect to the first and second wavenumber components. The successful alternative is to introduce circular polar coordinates (ρ, ϕ, z), defined by

$$X = \rho\cos\phi, \quad Y = \rho\sin\phi, \quad Z = z,$$

since then

$$(\boldsymbol{k}\times\boldsymbol{R})\cdot\boldsymbol{\nabla}_R = \rho\hat{\boldsymbol{\phi}}\cdot\boldsymbol{\nabla} = \frac{\partial}{\partial\phi},$$

with $\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}}\,\partial/\partial\rho + \hat{\boldsymbol{\phi}}\rho^{-1}\,\partial/\partial\phi + \boldsymbol{k}\,\partial/\partial z$, wherein $(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \boldsymbol{k})$ are appropriate unit vectors. This prevents the origin being moved from the axis of rotation to the point singularity, which disadvantage is to be anticipated on physical grounds since the well-known wake structure behind a sphere in uniform motion (Oseen 1927) suggests the existence of a corresponding circular wake effect here.

Write $\boldsymbol{Q}_0 = (Q_\rho, Q_\phi, Q_z)$ and introduce the Fourier transforms

$$Q_z = \frac{2}{\pi}\int_0^\infty \hat{Q}_z \sin\lambda z\,\mathrm{d}\lambda, \quad \begin{Bmatrix} Q_\rho \\ Q_\phi \\ P_0 \end{Bmatrix} = \frac{2}{\pi}\int_0^\infty \begin{Bmatrix} \hat{Q}_\rho \\ \hat{Q}_\phi \\ \hat{P} \end{Bmatrix} \cos\lambda z\,\mathrm{d}\lambda, \tag{4.12a, b}$$

$$\delta(z) = \frac{1}{\pi}\int_0^\infty \cos\lambda z\,\mathrm{d}\lambda, \tag{4.12c}$$

and, subsequently, the Fourier series

$$\hat{P} = \sum_{m=-\infty}^{\infty} \hat{P}_m\,\mathrm{e}^{\mathrm{i}m\phi}, \quad \hat{\boldsymbol{Q}} = \sum_{m=-\infty}^{\infty} \hat{\boldsymbol{Q}}_m\,\mathrm{e}^{\mathrm{i}m\phi}, \tag{4.13a, b}$$

$$2\pi\overset{\circ}{\delta}(\phi) = 1 + 2\sum_{m=1}^{\infty}\cos m\phi = \sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}m\phi},$$

where $\overset{\circ}{\delta}(\phi)$ is the periodic δ function, defined by

$$\overset{\circ}{\delta}(\phi) = \sum_{n=-\infty}^{\infty}\delta(\phi - 2n\pi).$$

Then the continuity equation (4.10) and the three components of the outer-field equation (4.9) reduce to the following set of ordinary differential equations, valid for all integers m:

$$\left(\frac{\mathrm{d}}{\mathrm{d}\rho}+\frac{1}{\rho}\right)\hat{Q}_{m\rho} + \frac{\mathrm{i}m}{\rho}\hat{Q}_{m\phi} + \lambda\hat{Q}_{mz} = 0, \tag{4.14a}$$

$$(L_{1,\,m} + \mathrm{i}m)\,\hat{Q}_{mz} + \lambda\hat{P}_m = 0, \tag{4.14b}$$

$$[L_{1,\,m\pm1} + \mathrm{i}(m\mp1)]\,(\hat{Q}_{m\rho} \pm \mathrm{i}\hat{Q}_{m\phi}) - \left(\frac{\mathrm{d}}{\mathrm{d}\rho}\mp\frac{m}{\rho}\right)\hat{P}_m = \mp\frac{3\mathrm{i}}{2B}\,\delta(\rho - B), \tag{4.15a, b}$$

where

$$L_{1,\,m} = \frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} - \left(\frac{m^2}{\rho^2}+\lambda^2\right). \tag{4.16}$$

According to Sneddon (1972), the Hankel transform $\mathscr{H}_m[f(\rho);\xi]$, defined for integers $m \geqslant 0$ by

$$\mathscr{H}_m[f(\rho);\xi] = \int_0^\infty \rho f(\rho)\, J_m(\xi\rho)\,\mathrm{d}\rho, \tag{4.17}$$

possesses the inversion formula

$$f(\rho) = \int_0^\infty \xi \mathscr{H}_m[f(\rho);\xi]\, J_m(\rho\xi)\,\mathrm{d}\xi, \tag{4.18}$$

and the properties

$$\mathscr{H}_m[L_{1,\,m} f;\xi] = -(\xi^2+\lambda^2)\,\mathscr{H}_m[f;\xi], \tag{4.19a}$$

$$\mathscr{H}_m\left[\frac{\mathrm{d}f}{\mathrm{d}\rho}+\frac{m+1}{\rho}f;\xi\right] = \xi\mathscr{H}_{m+1}[f;\xi], \tag{4.19b}$$

$$\mathscr{H}_m\left[\frac{\mathrm{d}f}{\mathrm{d}\rho}-\frac{m-1}{\rho}f;\xi\right] = -\xi\mathscr{H}_{m-1}[f;\xi]. \tag{4.19c}$$

Thus, when Hankel transforms of orders m and $(m\pm 1))$ are applied to equations (4.14) and the pair (4.15) respectively, the resulting set of equations obtained for each integer m has (by use of (4.19)) the matrix form

$$\begin{pmatrix} \lambda & \frac{1}{2}\xi & -\frac{1}{2}\xi & 0 \\ \mathrm{i}m-\lambda^2-\xi^2 & 0 & 0 & \lambda \\ 0 & \mathrm{i}(m-1)-\lambda^2-\xi^2 & 0 & \xi \\ 0 & 0 & \mathrm{i}(m+1)-\lambda^2-\xi^2 & -\xi \end{pmatrix}\begin{pmatrix} \mathscr{H}_m[\hat{Q}_{mz}] \\ \mathscr{H}_{m+1}[\hat{Q}_{m\rho}+\mathrm{i}\hat{Q}_{m\phi}] \\ \mathscr{H}_{m-1}[\hat{Q}_{m\rho}-\mathrm{i}\hat{Q}_{m\phi}] \\ \mathscr{H}_m[\hat{P}_m] \end{pmatrix}$$

$$= \tfrac{3}{2}\mathrm{i}\begin{pmatrix} 0 \\ 0 \\ -J_{m+1}(B\xi) \\ J_{m-1}(B\xi) \end{pmatrix}. \tag{4.20}$$

With

$$\Delta_m = (\lambda^2+\xi^2)(\lambda^2+\xi^2-\mathrm{i}m)^2+\lambda^2 \tag{4.21}$$

the determinant of the coefficients, the solutions of (4.20) are

$$(\lambda^2+\xi^2-\mathrm{i}m)^{-1}\mathscr{H}_m[\hat{P}_m;\xi] = \xi^{-1}\mathscr{H}_m[\hat{Q}_{mz};\xi] \tag{4.22a}$$

$$= -\frac{3\mathrm{i}}{4\Delta_m}\{(\lambda^2+\xi^2-\mathrm{i}m)[J_{m-1}(B\xi)+J_{m+1}(B\xi)]+\mathrm{i}[J_{m-1}(B\xi)-J_{m+1}(B\xi)]\} \tag{4.22b}$$

and

$$\mathscr{H}_{m\pm1}[\hat{Q}_{m\rho}\pm\mathrm{i}\hat{Q}_{m\phi};\xi] = -\frac{3\mathrm{i}}{4\Delta_m}\{(\lambda^2+\xi^2-\mathrm{i}m)\,\xi^2[J_{m-1}(B\xi)-J_{m+1}(B\xi)]$$
$$\mp 2[\lambda^2+\xi^2-\mathrm{i}(m\pm1)]\,\lambda^2 J_{m\pm1}(B\xi)\}. \tag{4.22c, d}$$

The inversion formula (4.18) then yields, on substitution into (4.13),

$$\hat{Q}_z = -\tfrac{3}{4}\mathrm{i}\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}m\phi}\int_0^\infty \frac{\xi^2 J_m(\rho x)}{\Delta_m}\{(\lambda^2+\xi^2-\mathrm{i}m)[J_{m-1}(B\xi)+J_{m+1}(B\xi)]$$
$$+\mathrm{i}[J_{m-1}(B\xi)-J_{m+1}(B\xi)]\}\,\mathrm{d}\xi, \tag{4.23a}$$

$$\hat{Q}_\rho+\mathrm{i}\hat{Q}_\phi = -\tfrac{3}{4}\mathrm{i}\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}m\phi}\int_0^\infty \frac{\xi J_{m+1}(\rho\xi)}{\Delta_m}\{(\lambda^2+\xi^2-\mathrm{i}m)\,\xi^2[J_{m-1}(B\xi)-J_{m+1}(B\xi)]$$
$$-2[\lambda^2+\xi^2-\mathrm{i}(m+1)]\,\lambda^2 J_{m+1}(B\xi)\}\,\mathrm{d}\xi. \tag{4.23b}$$

As the corresponding expression derived from (4.22*c, d*) for $(\hat{Q}_\rho-\mathrm{i}\hat{Q}_\phi)$ is merely the conjugate of (4.23*b*), it is therefore superfluous.

Similarly, the Fourier transform of the z-component of vorticity

$$\omega_z = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho Q_\phi) - \frac{\partial Q_\rho}{\partial \phi}\right] = \frac{2}{\pi}\int_0^\infty \hat{\omega}_z \cos \lambda z \, \mathrm{d}\lambda,$$

is given by

$$\hat{\omega}_z = -\frac{3}{4}\sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m\phi}\int_0^\infty \frac{\xi^2 J_m(\rho\xi)}{\Delta_m}\{\mathrm{i}\lambda^2[J_{m-1}(B\xi)+J_{m+1}(B\xi)]$$
$$+(\lambda^2+\xi^2)(\lambda^2+\xi^2-\mathrm{i}m)[J_{m-1}(B\xi)-J_{m+1}(B\xi)]\}\,\mathrm{d}\xi. \quad (4.24)$$

Although the ξ-integrals in (4.23) and (4.24) cannot be evaluated explicitly, the appearance in the integrands of products of Bessel functions of arguments $\rho\xi$ and $B\xi$ strongly suggest enhanced values at $\rho = B$ due to some form of circular wake (in accordance with physical expectations). The effect of this 'wake' upon the hydrodynamic force and torque exerted by the fluid on the sphere will be evaluated to order $T^{\frac{1}{2}}$ in the next section.

5. Force and torque on the sphere

Since Q_z and, hence, the ρ- and ϕ-components of vorticity vanish at $z = 0$, the hydrodynamic force $\boldsymbol{F}' = -\rho\nu a\Omega b\boldsymbol{F}$ and torque $\boldsymbol{M}' = -\rho\nu\Omega a^3 M\boldsymbol{k}$ (about the sphere centre) exerted on the sphere by the fluid are given by (Herron *et al.* 1975)†

$$\boldsymbol{F} = 6\pi[\boldsymbol{j} - T^{\frac{1}{2}}[(Q_\rho - Q_\rho^{(\mathrm{S})})_{(B,0,0)}\boldsymbol{i} + (Q_\phi - Q_\phi^{(\mathrm{S})})_{(B,0,0)}\boldsymbol{j}] + O(T)], \quad (5.1a)$$

$$M = 8\pi[1 - \tfrac{1}{2}T^{\frac{1}{2}}(\omega_z - \omega_z^{(\mathrm{S})})_{(B,0,0)} + O(T)]. \quad (5.1b)$$

Here, $\boldsymbol{Q}_0^{(\mathrm{S})}$ denotes the Stokes solution [cf. (3.20), (3.21)], i.e. that for which the Coriolis and inertia terms are omitted from (4.9), and therefore for which no imaginary terms appear in the coefficient matrix of the Hankel transforms.

Let $\boldsymbol{Q}_0^{(\mathrm{H})}$ denote the solution constructed by Herron *et al.* (1975), i.e. that for which only the inertia term is omitted from (4.9). Since those authors showed that

$$(Q_\rho^{(\mathrm{H})} - Q_\rho^{(\mathrm{S})})_{(B,0,0)} = \tfrac{3}{5\sqrt{2}}, \quad (Q_\phi^{(\mathrm{H})} - Q_\phi^{(\mathrm{S})})_{(B,0,0)} = -\tfrac{5}{7\sqrt{2}}, \quad (\omega_z^{(\mathrm{H})} - \omega_z^{(\mathrm{S})})_{(B,0,0)} = 0,$$

and since evidently $(\omega_z^{(\mathrm{H})})_{(B,0,0)} = 0$, it follows that the hydrodynamic force and torque formulas (5.1) can be rewritten as

$$\boldsymbol{F} = 6\pi\{\boldsymbol{j} - T^{\frac{1}{2}}[\tfrac{1}{\sqrt{2}}(\tfrac{3}{5}\boldsymbol{i} - \tfrac{5}{7}\boldsymbol{j}) + (Q_\rho - Q_\rho^{(\mathrm{H})})_{(B,0,0)}\boldsymbol{i} + (Q_\phi - Q_\phi^{(\mathrm{H})})_{(B,0,0)}\boldsymbol{j}] + O(T)\}, \quad (5.2a)$$

$$M = 8\pi[1 - \tfrac{1}{2}T^{\frac{1}{2}}(\omega_z)_{(B,0,0)} + O(T)]. \quad (5.2b)$$

In this way the calculation is reduced to that of finding the effects of including the inertia terms in (4.9).

After factoring the denominator Δ_m [cf. (4.21)] appearing therein, the m-summations of (4.23b) and (4.24) adopt the generic form

$$\sum_{m=-\infty}^{\infty} \frac{s_m\,\mathrm{e}^{\mathrm{i}m\phi}}{\lambda^2+\xi^2-\mathrm{i}m-\mathrm{i}\lambda(\lambda^2+\xi^2)^{-\frac{1}{2}}} = S(\phi), \quad (5.3)$$

where the summations

$$\sum_{m=-\infty}^{\infty} s_m\,\mathrm{e}^{\mathrm{i}m\phi} = s(\phi) \quad (5.4)$$

† Note that the Herron *et al.* (1975) definition of the Taylor number is exactly twice that used by us in (3.3). This difference arises because their fluid and, hence axes, rotate at infinity.

are combinations of the following summations (derived from formula 8.530.2 of Gradshteyn & Ryzhik 1980):

$$\sum_{m=-\infty}^{\infty} J_{m+1}(\rho\xi)\,J_{m+1}(B\xi)\,\mathrm{e}^{\mathrm{i}m\phi} = J_0[\xi(\rho^2+B^2-2\rho B\cos\phi)^{\frac{1}{2}}]\,\mathrm{e}^{-\mathrm{i}\phi}, \tag{5.5a}$$

$$\sum_{m=-\infty}^{\infty} J_{m+1}(\rho\xi)\,J_{m-1}(B\xi)\,\mathrm{e}^{\mathrm{i}m\phi} = \left(\frac{\rho\,\mathrm{e}^{\mathrm{i}\phi}-B}{\rho-B\,\mathrm{e}^{\mathrm{i}\phi}}\right) J_2[\xi(\rho^2+B^2-2\rho B\cos\phi)^{\frac{1}{2}}], \tag{5.5b}$$

$$\sum_{m=-\infty}^{\infty} J_m(\rho\xi)\,[J_{m-1}(B\xi)\pm J_{m+1}(B\xi)]\,\mathrm{e}^{\mathrm{i}m\phi} = \left(\begin{matrix}\mathrm{i}\rho\sin\phi\\ \rho\cos\phi-B\end{matrix}\right)\frac{2J_1[\xi(\rho^2+B^2-2\rho B\cos\phi)^{\frac{1}{2}}]}{(\rho^2+B^2-2\rho B\cos\phi)^{\frac{1}{2}}}. \tag{5.5c,d}$$

The subsequent algebraic expressions can be simplified by replacing the non-negative variables (λ, ξ) by the 'polar' coordinates $\chi(\geqslant 0)$ and $\eta(0 \leqslant \eta \leqslant 1)$, defined by

$$\lambda = \chi^{\frac{1}{2}}\eta, \quad \xi = \chi^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}. \tag{5.6a,b}$$

Then, since $S(\phi)$, defined by (5.3), satisfies the differential equation

$$\frac{\mathrm{d}S}{\mathrm{d}\phi}-(\chi-\mathrm{i}\eta)\,S = -s(\phi),$$

it may be readily shown that the summations required in (5.2) are of the form

$$S(0)-\frac{s(0)}{\chi-\mathrm{i}\eta} = [1-\mathrm{e}^{-2\pi(\chi-\mathrm{i}\eta)}]^{-1}\int_0^{2\pi}[s(\psi)-s(0)]\,\mathrm{e}^{-(\chi-\mathrm{i}\eta)\psi}\,\mathrm{d}\psi.$$

Hence

$$2(\lambda^2+\xi^2)\sum_{m=-\infty}^{\infty} s_m\left(\frac{\lambda^2+\xi^2-\mathrm{i}m}{\Delta_m}-\frac{\lambda^2+\xi^2}{\Delta_0}\right)$$
$$= [\cosh(2\pi\chi)-\cos(2\pi\eta)]^{-1}\int_0^{2\pi}[s(\psi)-s(0)]$$
$$\times\{\mathrm{e}^{(2\pi-\psi)\chi}\cos(\psi\eta)-\mathrm{e}^{-\psi\chi}\cos[(2\pi-\psi)\eta]\}\,\mathrm{d}\psi, \tag{5.7a}$$

and

$$2\lambda(\lambda^2+\xi^2)^{\frac{1}{2}}\sum_{m=-\infty}^{\infty} s_m\left(\frac{1}{\Delta_m}-\frac{1}{\Delta_0}\right)$$
$$= [\cosh(2\pi\chi)-\cos(2\pi\eta)]^{-1}\int_0^{2\pi}[s(\psi)-s(0)]$$
$$\times\{\mathrm{e}^{(2\pi-\psi)\chi}\sin(\psi\eta)+\mathrm{e}^{-\psi\chi}\sin[(2\pi-\psi)\eta]\}\,\mathrm{d}\psi, \tag{5.7b}$$

where $s(\psi)$ and (χ,η) are given by (5.4) and (5.6) respectively.

When (4.23*b*) is substituted into the Fourier transforms (4.12), the variables of integration changed according to (5.6), and the summations evaluated by means of (5.5*a*, *b*) and (5.7*a*, *b*), it follows that the quantities required to complete the calculation of $\boldsymbol{F}$ in (5.2*a*) are given by

$$(Q_\rho-Q_\rho^{(\mathrm{H})})_{(B,0,0)} = \frac{3}{4\pi}\int_0^\infty\int_0^1\frac{\chi^{\frac{1}{2}}\,\mathrm{d}\chi\,\mathrm{d}\eta}{\cosh(2\pi\chi)-\cos(2\pi\eta)}$$
$$\times\left[2\eta\int_0^{2\pi}\{J_0[2B\chi^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}\sin\tfrac{1}{2}\psi]\cos\psi-1\}\cosh[(2\pi-\psi)\chi]\sin(\psi\eta)\,\mathrm{d}\psi\right.$$
$$\left.+(1+\eta^2)\int_0^{2\pi}J_0[2B\chi^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}\sin\tfrac{1}{2}\psi]\sin\psi\cosh[(2\pi-\psi)\chi]\cos(\psi\eta)\,\mathrm{d}\psi\right], \tag{5.8a}$$

B	C_x	C_y	C_z	C
(0)	$-6/7$	$6/7$	0	∞
0.25	-0.851	0.851	-0.035	9.08
0.50	-0.838	0.849	-0.068	4.53
0.75	-0.817	0.847	-0.096	3.01
1.00	-0.790	0.846	-0.119	2.26
1.25	-0.759	0.848	-0.134	1.81
1.50	-0.725	0.853	-0.143	1.52
2.00	-0.653	0.879	-0.145	1.17
2.50	-0.585	0.922	-0.133	0.984
3.00	-0.525	0.978	-0.116	0.870
4.00	-0.432	1.096	-0.081	0.731
5.00	-0.366	1.196	-0.054	0.638
6.00	-0.320	1.270	-0.033	0.565
(∞)	0	—	—	—

TABLE 1. Numerical values of the force, torque and normalized Reynolds-number drag coefficients appearing in Eqs. (5.10), (5.11) and (6.3) for admissible values of B. (The extreme values $B = 0$ and ∞ are inadmissible. Where known, they are included here only for completeness.)

$$(Q_\phi - Q_\phi^{(\mathrm{H})})_{(B,\,0,\,0)} = \frac{3}{4\pi}\int_0^\infty\int_0^1 \frac{\chi^{\frac{1}{2}}\,\mathrm{d}\chi\,\mathrm{d}\eta}{\cosh(2\pi\chi)-\cos(2\pi\eta)}$$
$$\times\left[-2\eta\int_0^{2\pi} J_0[2B\chi^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}\sin\tfrac{1}{2}\psi]\,\sin\psi\,\sinh[(2\pi-\psi)\chi]\sin(\psi\eta)\,\mathrm{d}\psi\right.$$
$$+\int_0^{2\pi}\langle(1+\eta^2)\{J_0[2B\chi^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}\sin\tfrac{1}{2}\psi]\cos\psi-1\}$$
$$\left.+(1-\eta^2)\,J_2[2B\chi^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}\sin\tfrac{1}{2}\psi]\rangle\,\sinh[(2\pi-\psi)\chi]\cos(\psi\eta)\,\mathrm{d}\psi\right]. \quad (5.8b)$$

Similarly, but without the need to include $s(0)$ in (5.7a, b), (4.24) implies that the coefficient of the order-$T^{\frac{1}{2}}$ correction to the dimensionless torque M in (5.2b) is given by

$$\tfrac{1}{2}(\omega_z)_{(B,\,0,\,0)} = \frac{3}{4\pi}\int_0^\infty\int_0^1 \frac{\chi(1-\eta^2)^{\frac{1}{2}}\,\mathrm{d}\chi\,\mathrm{d}\eta}{\cosh(2\pi\chi)-\cos(2\pi\eta)}\times\int_0^{2\pi} J_1[2B\chi^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}\sin\tfrac{1}{2}\psi]$$
$$\times\{\eta\cos\tfrac{1}{2}\psi\,\sinh[(2\pi-\psi)\chi]\sin(\psi\eta)+\sin\tfrac{1}{2}\psi\,\sinh[(2\pi-\psi)\chi]\cos(\psi\eta)\}\,\mathrm{d}\psi. \quad (5.8c)$$

Thus the Fourier series at $\rho = B$, $\phi = 0$ have been expressed as ψ-integrals of simpler form.

Values of the triple integrals in (5.8a, b, c) have been computed numerically. In turn, these have been used to compute the B-dependent force and torque coefficients,

$$C_x = -\frac{3}{5\sqrt{2}} - [Q_\rho - Q_\rho^{(\mathrm{H})}]_{(B,\,0,\,0)}, \quad (5.9a)$$

$$C_y = \frac{5}{7\sqrt{2}} - [Q_\phi - Q_\phi^{(\mathrm{H})}]_{(B,\,0,\,0)}, \quad (5.9b)$$

and

$$C_z = -\tfrac{1}{2}(\omega_z)_{(B,\,0,\,0)}, \quad (5.9c)$$

required in the formulas (cf. (5.2a, b))

$$\boldsymbol{F} = 6\pi[\boldsymbol{j} + T^{\frac{1}{2}}(\boldsymbol{i}C_x + \boldsymbol{j}C_y) + O(T)], \tag{5.10}$$

$$M = 8\pi[1 + T^{\frac{1}{2}}C_z + O(T)]. \tag{5.11}$$

These coefficients are tabulated in table 1.

6. Discussion

6.1. *Rectilinear limit?*

In the limit, one would expect to recover the classical Oseen (1927)–Proudman & Pearson (1957) rectilinear Stokes-law drag correction coefficient,

$$\frac{F}{F^{(S)}} = 1 + \tfrac{3}{8}Re \equiv 1 + (\tfrac{3}{8}B)\,T^{\frac{1}{2}}, \tag{6.1}$$

from our curvilinear analysis by allowing the tether radius b to tend to infinity while, simultaneously, allowing the angular velocity Ω to tend to zero in such a way that the product $\Omega b (\equiv U)$ remains finite. For in this limit (and with $a/b \ll 1$), the Taylor number $T = \Omega^2 a/v$ vanishes from the governing equations, whereas the Reynolds number $Re = Ua/v$ remains. According to (3.9) or (3.10), this asymptotic situation corresponds to the limiting case where $B \gg 1$. Were this intuitive guess sustained, the C_y values tabulated in table 1 would tend to $\frac{3}{8}B$ as required by (5.10) and (6.1). While our numerics do not extend beyond the last entry, $B = 6$, the data appear to preclude this possibility. Indeed, the values tabulated in the last column of table 1 of the normalized coefficient

$$C \overset{\text{def}}{=} \frac{8C_y}{3B}, \tag{6.2}$$

appearing in the sphere drag formula

$$\frac{F}{F^{(S)}} = 1 + C\tfrac{3}{8}Re, \tag{6.3}$$

seem to be tending towards the limiting value $C = 0$ as $B \to \infty$, rather than towards the value $C = 1$ that would be required to achieve the Oseen limit (6.1).

The explanation of this apparent paradox becomes clear upon using (3.9) to rewrite the basic inner and outer equations explicitly in terms of Re and B, rather than T and B. In terms of the former pair of parameters, the exact governing equations become (with prime affix denoting the correspondingly numbered equation previously written in the T and B format):

Inner

$$\frac{Re^2}{B^2}[\boldsymbol{k} \times \boldsymbol{q} - (\boldsymbol{k} \times \boldsymbol{r}) \cdot \nabla \boldsymbol{q}] + Re\,(\boldsymbol{q} - \boldsymbol{j}) \cdot \nabla \boldsymbol{q} = -\nabla p + \nabla^2 \boldsymbol{q}, \tag{3.12$'$}$$

$$\boldsymbol{q} = \boldsymbol{j} + \frac{Re}{B^2}\boldsymbol{k} \times \boldsymbol{r} \quad \text{on } |\boldsymbol{r}| = 1, \tag{3.13$'$}$$

along with (3.6) and (3.7);

Outer

$$\boldsymbol{k} \times \boldsymbol{Q} + Re\,\boldsymbol{Q} \cdot \nabla_R \boldsymbol{Q} - (\boldsymbol{k} \times \boldsymbol{R}) \cdot \nabla_R \boldsymbol{Q} = -\nabla_R P + \nabla_R^2 \boldsymbol{Q}, \tag{4.2$'$}$$

$$\boldsymbol{Q} = \frac{1}{Re}\boldsymbol{k} \times \boldsymbol{R} \quad \text{on } |\boldsymbol{R} - \boldsymbol{i}B| = 1, \tag{4.5$'$}$$

along with (4.3) and (4.4).

Whereas the $B \to \infty$ limit in the inner equations causes them to properly reduce to the corresponding Proudman & Pearson (1957) inner equations, the same is not true of the outer equations – for which the curvilinear terms $\boldsymbol{k} \times \boldsymbol{Q}$ and $-(\boldsymbol{k} \times \boldsymbol{R}) \cdot \nabla_R \boldsymbol{Q}$ remain, despite the Taylor number being identically zero. Accordingly, our drag result cannot be expected to approach the limiting value (6.1). Indeed, the $B \to \infty$ limit is inadmissible in our analysis, since only B-values of order unity are consistent with our scaling hypothesis (3.9) and choice of origin. And both of these differ from what would be required to derive the rectilinear trajectory limit, for which the sphere never encounters its own wake.†

6.2. *Antisedimentation Tube*

Though the preceding analysis has been presented in the guise of a non-axial rotating sphere in an otherwise stationary fluid, the actual motivation for the present study involved the reciprocal situation of a non-axial stationary sphere in a rotating fluid. The latter conditions represent the physical circumstances prevailing in an 'antisedimentation' tube (Dill & Brenner 1983; Nadim, Cox & Brenner 1985) currently under development. This advice (figure 2) permits a non-neutrally buoyant sphere to remain permanently suspended against the force of gravity by balancing its downward settling motion against an upward fluid current created by the steady rotation of a fluid-filled circular cylinder rotating about a horizontal axis. Until now, calculations pertaining to the operation of this device have only been available for the Stokes-flow limit, where inertial, centrifugal and Coriolis forces are all negligible.

Relative to a stationary observer, the dimensional equations of motion governing the steady fluid motion depicted in figure 2 are

$$\boldsymbol{v}' \cdot \nabla' \boldsymbol{v}' = -\frac{1}{\rho} \nabla' \pi' + \nu \nabla'^2 \boldsymbol{v}',$$

$$\nabla' \cdot \boldsymbol{v}' = 0,$$

$$\boldsymbol{v}' = \boldsymbol{0} \quad \text{at } |\boldsymbol{r}'| = a,$$

$$\boldsymbol{v}' \to \boldsymbol{\Omega} \times \boldsymbol{R}' \quad \text{as } |\boldsymbol{R}'| \to \infty.$$

Gravity forces acting upon the incompressible fluid have been absorbed into the pressure term. Define

$$\boldsymbol{q}' = \boldsymbol{v}' - \boldsymbol{\Omega} \times \boldsymbol{R}'$$

and

$$p' = \pi' + \tfrac{1}{2}[\Omega^2 R'^2 - (\boldsymbol{\Omega} \cdot \boldsymbol{R}')^2].$$

The fields $(\boldsymbol{q}', p')$ represent the fluid motion as seen by a rotating observer.

With $\boldsymbol{r}'$ defined as in (2.9), we follow (3.1) in introducing the non-dimensional variables

$$\boldsymbol{r}' = a\boldsymbol{r}, \quad \nabla' = a^{-1}\nabla,$$

$$\boldsymbol{q}' = \overline{U}\boldsymbol{q}, \quad p' - p'_\infty = \rho \nu a^{-1} \overline{U} p,$$

† It is interesting to note in this wake-interference context that the classical Oseen (1927) analysis (cf. Happel & Brenner 1983, p. 281) for *two* (relatively distant) equal spheres translating with identical velocities along their line of centres reveals that the trailing sphere experiences a smaller drag force than does the leading one. This asymmetric 'shielding' effect is experimentally well documented (Happel & Pfeffer 1960), in the sense that when two proximate spheres settle slowly in a viscous fluid (in the Reynolds-number range where inertial effects are sensible), the trailing one always overtakes the leading one.

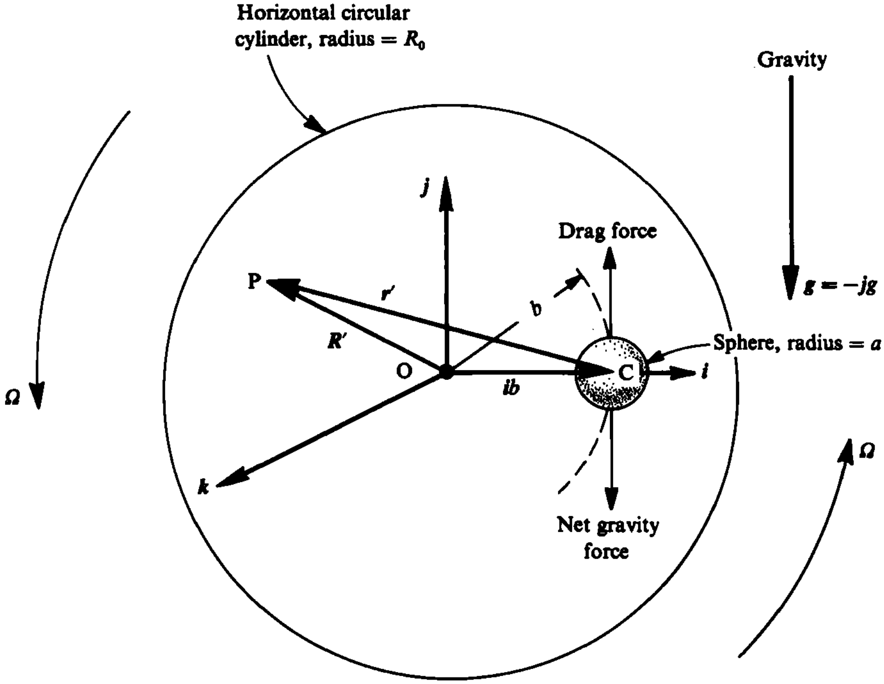

FIGURE 2. Antisedimentation tube. A homogeneous non-neutrally buoyant sphere of radius a, whose density exceeds that of the fluid, can be prevented from sedimenting by counterclockwise rotation of the horizontal cylinder (and fluid contents) at an angular velocity $\boldsymbol{\Omega} = \boldsymbol{k}\Omega$ (with $\boldsymbol{k}$ out of the plane of the paper) just sufficient that the drag force exerted by the fluid on the sphere (in the $\boldsymbol{j}$-direction, opposite to gravity) exactly balances the net gravity force $-\boldsymbol{j}4\pi a^3|\Delta\rho|\,g/3$ on the sphere. Here, $|\Delta\rho|$ is the magnitude of the sphere-fluid density difference and g the acceleration due to gravity. For simplicity in our calculations the sphere is assumed tethered to the cylinder centre to prevent the centrifugal and other radially directed 'inertial' forces from moving the sphere outward, in the $\boldsymbol{i}$ direction. In such circumstances the sphere centre C will lie in the same horizontal plane as the cylinder axis O, perpendicular to gravity (Dill & Brenner 1983, Nadim *et al.* 1985), held by the tether at a fixed distance b from the cylinder axis. To bring the present problem into correspondence with the problem previously treated, it will be supposed that the sphere does not rotate (about an axis through its centre). It should be understood, however, that the corresponding *untethered* sphere would, in fact, rotate owing to the absence of any restraining couples. The angular velocity of this free rotation can eventually be determined by formulae of the type (5.11). It will also be assumed that $a/(R_0-b) \ll 1$, so that wall effects on the sphere are negligible.

with the velocity $\overline{U}$ yet to be chosen. Define T as in (3.3) (with $\Omega > 0$) and [cf. (3.4)]

$$\overline{Re} \overset{\text{def}}{=} \frac{\overline{U}a}{\nu} \ll 1.$$

In terms of these, the fields $(\boldsymbol{q}, p)$ are governed by the system of dimensionless equations

$$T\boldsymbol{k}\times\boldsymbol{q}+\overline{Re}\left[\boldsymbol{q}+\frac{\Omega b}{\overline{U}}\left(\boldsymbol{j}+\frac{a}{b}\,\boldsymbol{k}\times\boldsymbol{r}\right)\right]\cdot\nabla\boldsymbol{q} = -\nabla p+\nabla^2\boldsymbol{q},$$

$$\nabla\cdot\boldsymbol{q} = 0,$$

$$\boldsymbol{q}\to\boldsymbol{0} \quad \text{as } |\boldsymbol{r}|\to\infty,$$

$$\boldsymbol{q} = -\frac{\Omega b}{\overline{U}}\left(\boldsymbol{j}+\frac{a}{b}\,\boldsymbol{k}\times\boldsymbol{r}\right) \quad \text{on } |\boldsymbol{r}| = 1.$$

The choice $\overline{U} \overset{\text{def}}{=} -\Omega b$ renders this system of equations identical with (3.5)–(3.8), except that since $\overline{Re} = -Re = -\Omega ab/v$, the coefficient of Re [cf. (3.2) and (3.4)] in equation (3.5) is not negative rather than positive. Apart from this algebraic-sign difference the analysis is identical with that already given in the preceding sections. Indeed, the results for the present antisedimentation case can be trivially obtained by replacing b by $-b$ and, hence, B by $-B$ in the prior analysis. This sign change has an impact in two ways. First, at the beginning of §5, we now have that

$$\boldsymbol{F}' = \rho v a \Omega |b| \boldsymbol{F},$$

where $\boldsymbol{F}$ is the non-dimensional force appearing in (5.4*a*). [The corresponding relation between $\boldsymbol{M}'$ and $\boldsymbol{M}$, with $\boldsymbol{M}$ given as in (5.1*b*), remains unaltered however.] Secondly, since the Bessel functions J_m appearing in (5.8*a–c*) are respectively odd or even functions of their arguments according as the integer m is itself odd or even, it follows that the three coefficients defined in (5.9*a–c*) possess the following properties:

$$C_x(-B) = C_x(B), \quad C_y(-B) = C_y(B), \quad C_z(-B) = -C_z(B).$$

With this proviso the tabulation of table 1 can now be used to calculate $\boldsymbol{F}$ and $\boldsymbol{M}$ via (5.10) and (5.11) for B in the range $-\infty < B \leqslant 0$.

This extension of the analysis to the case $B < 0$ thus furnishes the requisite data required to analyse the antisedimentation tube of figure 2 for circumstances in which inertial effects, though small, are not negligible.

The boundary conditions for the original tethered-sphere problem are, from the viewpoint of a rotating observer, identical to those of the antisedimentation problem as seen by a stationary observer – namely zero velocity on the sphere surface and a velocity equal to $\boldsymbol{\Omega} \times \boldsymbol{R}'$ at infinity. Moreover, each problem is steady with respect to its respective observer. However, the differential equations governing the respective velocity fields are different, owing to the necessity for including fictitious body forces in the rotating-reference-frame problem. Accordingly, the hydrodynamic forces and couples on the sphere cannot generally be the same for both problems. While our calculations reveal this to be true of the couple, the forces are in fact the same, at least to the order of the approximation. This equality cannot, however, continue to persist at all others.

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